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MODELS FOR SIMULATING BMA AND LIBOR INTEREST RATES

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ABSTRACT: During the last 10 years, U.S. municipalities and not-for-profit organizations have issued more than $1 trillion in tax-exempt variable rate securities. Cash flows from these securities often closely track the Bond Market Association Municipal Swap Index (BMA). BMA itself tends to hold a relationship to the London Interbank Offered Rate (LIBOR) driven largely by top federal marginal tax rates. Motivated in part by the widespread use of LIBOR indexed derivatives to hedge tax-exempt variable rate bonds, this paper explores the historical BMA and LIBOR series and suggests methods for simulating these rates into the future for purposes of hedge analysis and general structuring.

1. NOTATIONS AND SET UP

Let $B(t)$ and $R(t)$ be BMA rate and 1-month Libor rates respectively. We will model the behavior of the logs of these rates

$$LB(t) = \ln (B(t)) \,,$$
$$LR(t) = \ln (R(t)).$$

A Principal Component Analysis (PCA) factor decomposition of the pair $LB(t), LR(t)$ gives the following model

$$LB(t) = L_0(LB) + L_1(LB)\, f_1(t) + L_1(LB)\, f_2(t) \,,$$
$$LR(t) = L_0(LR) + L_1(LR)\, f_1(t) + L_1(LR)\, f_2(t),$$

where

$$L_i = [L_i(LB), L_i(LR)]$$

are the constant factor loadings and $f_i(t)$ are the dynamic factors or factor scores.

For the ratio of the two rates we use the notation

$$\Theta(t) = \frac{B(t)}{R(t)} = \exp(LB(t) - LR(t)) \,,$$

so for log-ratios we have

$$\theta(t) = \ln(\Theta(t)) = LB(t) - LR(t) \,.$$

Finally we write the factor model for the log-ratio

$$\begin{aligned} \theta(t) &= (L_0(LB) - L_0(LR)) + (L_1(LB) - L_1(LR))\, f_1(t) \\ &\quad + (L_2(LB) - L_2(LR))\, f_2(t) \\ &= DL_0 + DL_1 f_1(LB) + DL_2 f_2. \qquad (1.1) \end{aligned}$$

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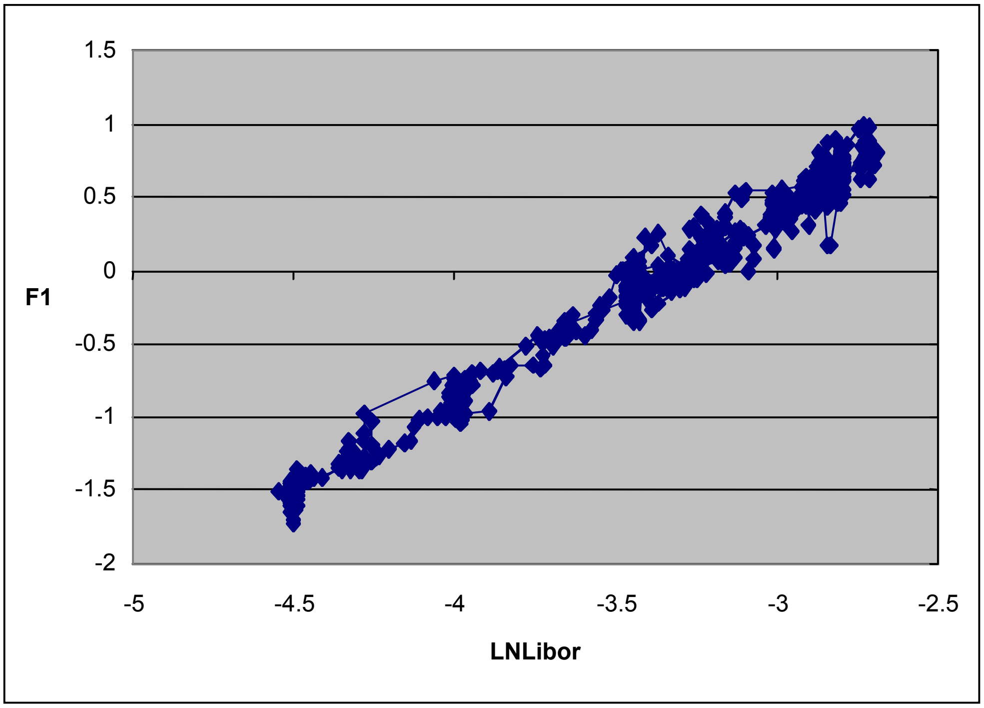
2. INITIAL DATA ANALYSIS

Using weekly BMA and 1-month Libor data to January 1, 1994 we estimate the following 2 factor model:

$$\begin{bmatrix} L_0(LB) \, L_1(LB) \, L_2(LB) \\ L_0(LR) \, L_1(LR) \, L_2(LR) \end{bmatrix} = \begin{bmatrix} -3.69 \, 0.64 \;\; 0.77 \\ -3.32 \, 0.77 \, -0.64 \end{bmatrix}$$



2.1. **Interpretation of the first factor**. After we decompose the PCA we note the similarity between f_1, (t) and $LR(t)$. The slope and the intercept of the regression relationships are

$$a = 1.28,$$
$$b = 4.26$$

and the regression model is

(2.1) $$f_1(t) = aLR(t) + b.$$

Therefore, we interpret the first factor as the Libor level itself.

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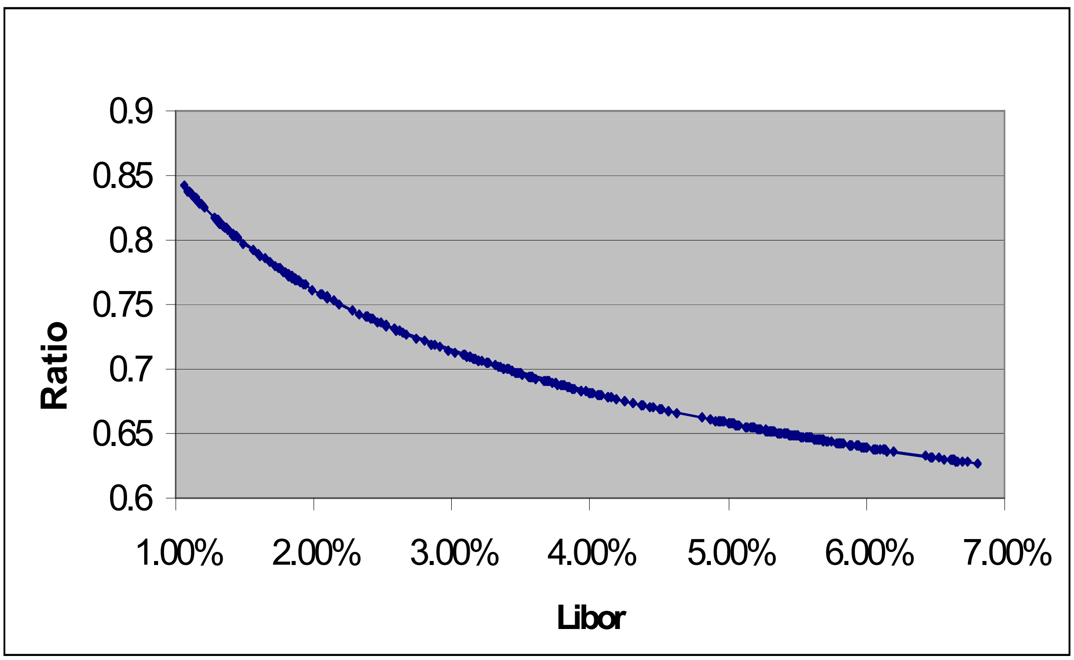
2.2. **BMA-Libor ratio and the rate compression effect**. With the factor loadings we look at the log-BMA-Libor ratio as a function of Libor level.

$$\begin{aligned}\theta(t) &= DL_0 + DL_1 f_1 + DL_2 f_2 \\ &= (DL_0 + DL_1 b)) + aDL_1 LR\,(t) + DL_2 f_2 \\ &= -0.895 - 0.159 LR\,(t) + 1.41\, f_2.\end{aligned}$$

And for the ratio itself we have

$$\Theta(t) = \exp(\theta(t)) = e^{-0.895-0.159\mathrm{LR}(t)+1.41f2}$$

$$= e^{-0.895} R(t)^{-0.159} e^{1.41f2}$$



As the following chart shows the model captures the rates compression effect which means that as the Libor rates level drops down the BMA/Libor tends to grow.

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3. SIMULATION PROCEDURE

We suggest organizing the joint simulation of Libor and BMA as follows:

(1) Simulate the random drivers. In the two factor version of the model the random drivers are $LR(t)$ and $f_2(t)$. So, during the first step we simulate the time series of $LR(t)$ and $f_2(t)$ for $t = t_0,\ldots,T$, where t_0 and T are the first and the last dates of the stimulated period. We will cover the simulation method in the following section.
(2) Calculate the first factor $f_1(t)$, $t = t_0,\ldots,T$. Use (2.1) for calculation.
(3) Calculate BMA-Libor log-spread $\theta(t)$. Use (1.1) for calculation.
(4) Calculate $R(t)$ **and** $\boldsymbol{\Theta}(t)$, $t = t_0, \ldots,T$. Just take exponents of $LR(t)$ and $\theta(t)$.
(5) Calculate $B(t)$, $t= t_0,\ldots, T$. The formula for calculation is

$$B(t) = R(t)\,\boldsymbol{\Theta}(t)\,.$$

4. SIMULATION OF THE DRIVERS

We would like to use a model for the random drivers simulation that generates mean reverting trajectories with seasonal effects.

We suggest using an integrated seasonal autoregressive moving average model class.

4.1. **Seasonal autoregressive model**. To define the seasonal autoregressive model SARI(p. P, d, q) we use the following notations.

Let B represent a one-day time shift operator. For example, if X_t, $t = -\infty,\ldots, -1, 0, 1,\ldots, \infty$, is a time sequence with integer time steps then by definition

$$DX_t - X_{t-1},$$

and

$$D^k X_t = X_{t-k}$$

Let

$$\ell(D) = 1 + \ell_1 D + \ldots + \ell_P D^P$$

be a polynomial operator (called autoregression operator) of order p such that

$$\begin{aligned}\ell(D)\,X_t &= (1 + \ell_1 D + \ldots + \ell_p D^p)\,X_t,\\ &= X_t + \ell_1 X_{t-1} + \ldots + \ell_p X_{t-p.}\end{aligned}$$

Similarly let

$$\Phi(D^s) = 1 + \ell_1 D^s + \ldots + \Phi_P D^{Ps}$$

be a polynomial operator (called seasonal autoregression operator) of order p such that

$$\begin{aligned}\Phi(D^s)\,X_t &= (1 + \Phi_1 D^s + \ldots + \Phi_P D^{Ps})\,X_t\\ &= X_t + \Phi_1 X_{t-s} \ldots \Phi_P X_{t-Ps.}\end{aligned}$$

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Also let

$$\psi(D) = 1 + \psi_1 D + \ldots + \psi_q D^q$$

be a polynomial operator (called moving average operator) of order q.
We call X_t a SARMA(P, p, q) (seasonal autoregression moving average of orders P, p, q) process if

$$\Phi(D^s)\,\ell(D)\,X_t = \mu + \psi(D)\,\varepsilon_t,$$

where μ is the model constant and ε_t is the white noise innovation sequence, i.e. sequence of independent identically distributed Gaussian random variables with zero mean and standard deviation σ

$$\varepsilon_t \sim N(0, \sigma)\,.$$

Finally we define ISARMA (P, p, q, d), an integrated seasonal autoregressive moving average process, if $\Delta^d X_t$ is a SARMA (P, p, q) process,

$$\Delta X_t = X_{t+1} - X_t,$$
$$\Delta^d X_t = \Delta^{d-1} X_t.$$

Example 1. *Let the seasonal period be $s = 13$, AR order be $p = 1$, seasonal AR order be $P = 4$, integration order $d = 1$, moving average order $q = 0$. Then ISARMA (4,1,0,1) process is by definition*

$$\Phi(D^s)\,\ell(D)\,\Delta X_t = \mu + \varepsilon_t$$

where

$$\begin{aligned}\ell(D)\,\Phi(D^s) &= (1 + \ell_1 D\, \Phi_1 D^s + \Phi_2 D^{2s} + \Phi_3 D^{3s} + \Phi_4 D^{4s}) \\ &= 1 + \ell_1 D + \Phi_1 D^s + \Phi_1 \ell_1 D^{s+1} + \Phi_2 D^{2s} + \Phi_2 \ell_1 D^{2s+1} \\ &\quad + \Phi_3 D^{3s} + \Phi_3 \ell_1 D^{3s+1} + \Phi_4 D^{4s} + \Phi_4 \ell_1 D^{4s+1}.\end{aligned}$$

Therefore, the model equation is

$$\begin{aligned}\mu + \varepsilon t &= (1 + \ell_1 D + \Phi_1 D^{13} + \Phi_1 \ell + D^{14} + \Phi_2 D^{26} + \Phi_2 \ell_1 D^{27} \\ &\quad + \Phi_3 D^{39} + \Phi_3 \ell + D^{40} + \Phi_4 D^{52} + \Phi_4 \ell_1 D^{53}\ \Delta X_t \\ &= \Delta X_t + \ell_1 \Delta X_{t-1} + \Phi_1 \Delta X_{t-13} + \Phi_1 \ell_1 \Delta X_{t-14} + \Phi_2 \Delta X_{t-26} + \Phi_2 \ell_1 \Delta X_{t-27} \\ &\quad + \Phi_3 \Delta X_{t-39} + \Phi_3 \ell_1 \Delta X_{t-40} + \Phi_4 \Delta X_{t-52} + \Phi_4 \ell_1 \Delta X_{t-53} \qquad (4.1)\end{aligned}$$

Example 2. *If now $P = 4$, $p = 0$, $q = 3$, $d = 0$ then the model ISARMA (4,0,3,0) is defined by operators*

$$\Phi(D^s) = (1 + \Phi_1 D^s + \Phi_2 D^{2s} + \Phi_3 D^{3s} + \Phi_4 D^{4s}),$$
$$\Psi(D) = (1 + \Psi_1 D + \Psi_2 D^2 + \psi_3 D^3)$$

Then the model equation is

$$\Phi(D^s)\,X_t = \mu + \Psi(D)\,\varepsilon_t$$

or

(4.2)

$$Xt + \Phi_1 X_{t-s} + \Phi_2 X_{t-2s} + \Phi_3 X_{t-3s} + \Phi_4 X_{t-4s} = \mu + \varepsilon t + \Psi_1 \varepsilon_{t-s} + \Psi_2 \varepsilon_{t-2} + \Psi_3 \varepsilon_{t-3.}$$

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4.2. **Simulation of integrated SAR processes**. Simulation of integrated SARMA processes is straightforward. One needs to solve the model equation for the term ΔX_t. For example, for the first of our examples we derive from (4.1)

$$\begin{aligned} \Delta X_t &= X_{t+1} - X_t \\ &= \mu - \ell_1 \Delta X_{t-1} - \Phi_1 \Delta X_{t-13} - \Phi_1 \ell_1 \Delta X_{t-14} - \Phi_2 \Delta X_{t-26} - \Phi_2 \ell_1 \Delta X_{t-27} \\ &\quad - \Phi_3 \Delta X_{t-39} - \Phi_3 \ell_1 \Delta X_{t-40} - \Phi_4 \Delta X_{t-52} - \Phi_4 \ell_1 \Delta X_{t-53} \\ &\quad + \varepsilon_t. \end{aligned}$$

Now assume that the model parameters are known and the process is observed up until time t. That means that all terms in the right hand part of the formula above, but ε_t are known at time t.

The last term ε_t is a normally distributed random variable with zero mean and known standard deviation σ.
Once ΔX_t is calculated we find X_{t+1} as

$$X_{t+1} = X_t + \Delta X_t.$$

The second example implies

$$X_t = \mu - \Phi_1 X_{t-s} - \Phi_2 X_{t-2s} - \Phi_3 X_{t-3s} - \Phi_4 X_{t-4s} + \varepsilon t + \Psi_1\, \varepsilon_{t-s} + \Psi_2\, \varepsilon_{t-2} + \Psi_3\, \varepsilon_{t-3.}$$

From this equation we calculate the process X_t at time *t* if we know previous values of X_t and previously simulated independent identically distributed variables ε_t.

Our data analyses showed that we can assume that log-Libor can be described the best by the model (4.1) and F2 is best simulated by (4.2).

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4.3**. Estimated parameters**. Given our sample we estimated the following model parameters for (4.1) and (4.2). The log-Libor rates process we simulate by (4.1) with

μ = –1.67E – 05,
σ = 0.000595,
s = 13.
P = 4,
Φ_1 = –0.058,
Φ_2 = –0.043,
Φ_3 = –0.0066,
Φ_4 = –0.1688,
p = 1
ℓ_1 = –0.1325,
q = 0,
d = 1.

The f_2 factor process we simulate by (4.2) with

μ = 0.0024.
σ =0.0045,
s = 13,
P = 4,
Φ_1 = –0.155,
Φ_2 = –0.203,
Φ_3 = –0.024,
Φ_4 = –0.307,
p = 0,
q = 3,
ψ_1 = 0.843.
ψ_2 = 0.452
ψ_3 = 0.172
d = 1

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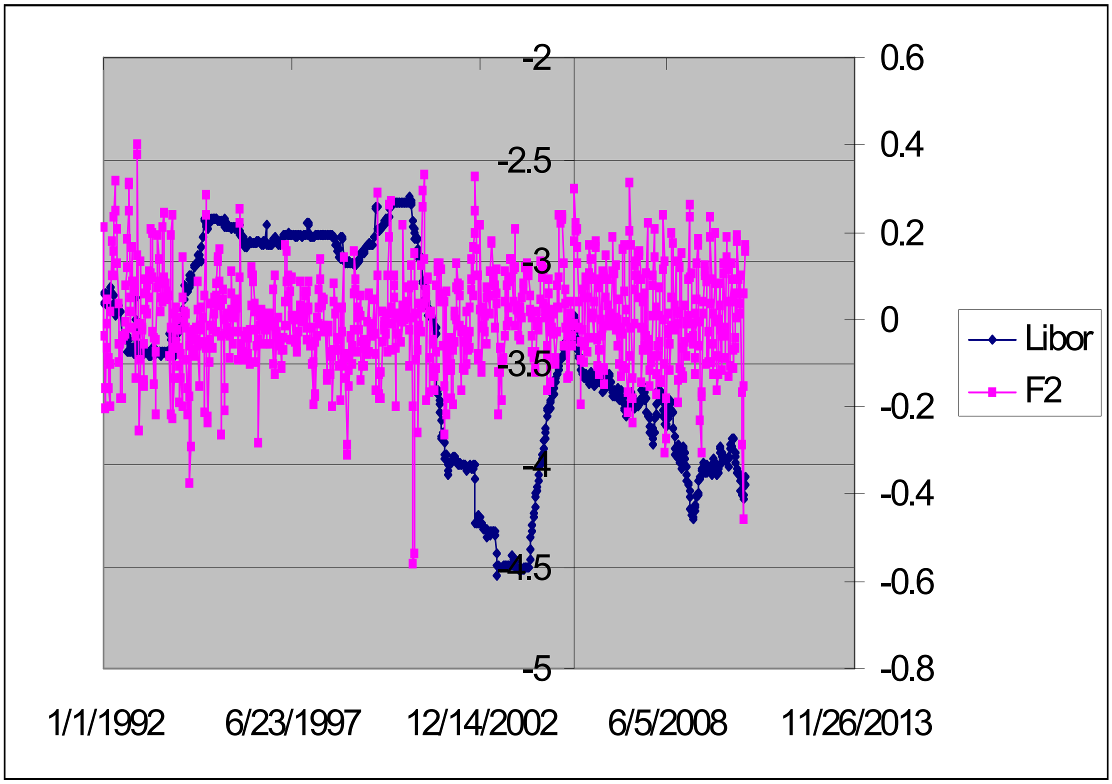
4.4. **Simulation results.** The following graph represents the simulation results of the two random drivers of the model. To the left of the Y-axis we see the observed history and to the right, one simulated trajectory.



And the next graph shows the simulated variables: Libor rates, BMA rates and the ratio. Again, to the left of the vertical axis we see the observed samples and to the right is the simulated future.

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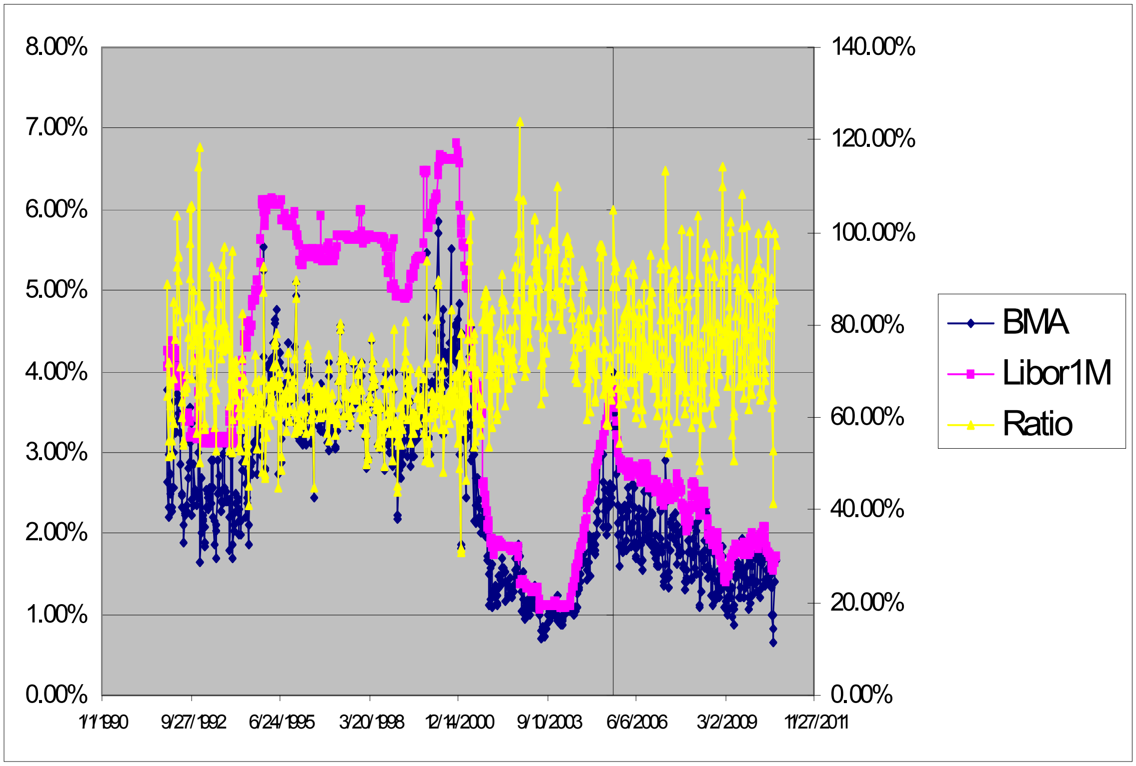

5. ONE-FACTOR VERSION

It is possible to use a simplified version of the approach based on one factor. In this case the model is parameterized as

$$LB(t) = L_0(LB) + L_1(LB) f_1(t),$$
$$f_1(t) = aLR(t) + b.$$

Then

$$\theta(t) = \ln(\Theta(t)) = LB(t) - LR(t)$$
$$= L_0(LB) + bL_1(LB) + (aL_1(LB) - 1)\, LR(t).$$

With this simplification one needs only to simulate the sequence LR (t) using (4.1). From that sequence calculate $\theta(t)$, then $R(t)$, $\Theta(t)$ and $B(t)$.

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6. CONCLUSION

Municipalities and not-for-profit issuers of tax-exempt floating rate debt frequently encounter decisions driven by the expected levels for and relationship between the BMA index and LIBOR. To that end, this paper analyzes the factors that have driven these rates historically and suggests methodologies for capturing some of those features in an interest rate model. We find a significant "compression" effect or negative correlation between the first factor (approximately LIBOR) and the second factor (approximately BMA/LIBOR) which as a result, given a static expectation for marginal tax rates, may be a desirable feature in a simulation model. Interestingly, we find that even a one factor PCA model can capture a significant degree of this compression effect. Autoregressive models are also used and as expected, we find a significant amount of seasonality in the second factor, largely coincident with seasonal tax effects. Depending upon the frequency of simulation intervals and purpose of the analysis, this may or may not be an important element to incorporate in a model.

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